

# Gluino condensation in an interacting instanton ensemble

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We perform a semiclassical study of chiral symmetry breaking and of the spectrum of the Dirac operator in QCD with adjoint fermions. For this purpose we calculate matrix elements of the adjoint Dirac operator between instanton zero modes and study their symmetry properties. We present simulations of the instanton ensemble for different numbers of Majorana fermions in the adjoint representation. These simulations provide evidence that instantons lead to gluino condensation in supersymmetric gluodynamics.

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## I. INTRODUCTION

In order to improve our understanding of non-perturbative phenomena in QCD it is useful to view QCD from a larger perspective, as a member of a family of QCD-like theories with different matter contents. In this context we would like to understand the phase structure of QCD-like theories with  $N_f$  fermions in the fundamental representation of the gauge group and  $N_{ad}$  fermions in the adjoint representation. Theories with adjoint fermions are special because the action may display a symmetry that connects bosonic and fermionic degrees of freedom, supersymmetry. Supersymmetry imposes powerful restrictions on the structure of the low energy effective action. These constraints have been used, for example, to determine the phase structure of  $N=1$  supersymmetric (SUSY) QCD with  $N_c$  colors and  $N_f$  flavors of fundamental quarks [1,2].

In addition to that, supersymmetry provides the opportunity to isolate certain non-perturbative effects, in particular instantons. This idea has been used in order to calculate the gluino condensate in the simplest supersymmetric gauge theory, SUSY gluodynamics. The strategy behind the so-called weak coupling instanton calculation [3–7] is to add to the theory a fundamental fermion together with its scalar superpartners and consider the regime where the expectation value of the scalar field is large. In this case there is a unique non-perturbative superpotential induced by instantons. Since the scalar vacuum expectation value (vev) is large, instantons are semi-classical and the superpotential can be calculated reliably. The superpotential determines the gluino condensate in the theory with additional matter. Finally, the matter fields can be decoupled by sending their mass to infinity. The result for the gluino condensate in  $SU(2)$  supersymmetric gluodynamics is

$$\langle \lambda \lambda \rangle = 6\Lambda^3, \quad (1)$$

where  $\Lambda$  is the scale parameter defined in [8]

$$\Lambda = M_{PV} \left( \frac{16\pi^2}{bg^2} \right)^{1/3} \exp \left( - \frac{8\pi^2}{bg^2} \right). \quad (2)$$

Here,  $b = 3N_c$  is the first coefficient of the beta function in supersymmetric gluodynamics and  $M_{PV}$  is a Pauli-Villars regulator.

There is an old puzzle concerning this result. The puzzle is connected with the fact that there is an alternative method for calculating the gluino condensate, usually referred to as the strong coupling instanton method [9–12]; for a review see [13]. In the case of  $N_c = 2$  one considers a four fermion correlation function. This correlation function is a topological quantity. Not only can it be saturated with one instanton, but supersymmetry implies that the correlator is just a constant. At short distance, one expects that this constant is saturated by small instantons and can be calculated reliably. The gluino condensate is then extracted by using clustering. The puzzle is that the result differs from the WCI calculation by a factor 4/5.

Several suggestions have been put forward in order to resolve the puzzle [14–17]. We do not wish to discuss these possibilities in detail. Instead, we would like to employ a somewhat different, more qualitative approach. Even though there is no direct instanton contribution to the gluino condensate, one would still expect configurations with instantons and anti-instantons to contribute to the gluino condensate. Here we have in mind that the theory is studied in a finite volume and in the presence of a non-zero mass term. The thermodynamic limit is approached by taking the volume to infinity before we let the mass go to zero. The mechanism for gluino condensation is similar to the instanton liquid picture of quark condensation in ordinary QCD [18–20]. For  $N_f > 1$  there is no direct instanton contribution to the quark condensate but chiral symmetry breaking may take place in an ensemble of instantons and anti-instantons in the thermodynamic limit. The Banks-Casher relation  $\langle \bar{q}q \rangle = -\pi\rho(0)$  [21] connects the quark condensate with the density of eigenvalues of the Dirac operator at zero virtuality. For simplicity let us consider an ensemble with an equal number of instantons and anti-instantons. In this case the Dirac operator no longer has any exact zero modes. However, if the interaction between the instantons is sufficiently weak, the approximate zero modes associated with individual instantons and anti-instantons form a zone around zero virtuality and lead to spontaneous chiral symmetry breaking. This quark condensation mechanism has been investigated in some de-

tail, both analytically and on the lattice [22,23], and the results seem to support the instanton picture.

In the present work we wish to extend these studies to theories with fermions in the adjoint representation. Since we are dealing with a strongly coupled theory, our calculations are necessarily approximate. In particular, we will have to restrict ourselves to the contribution of small instantons for which the semi-classical description is appropriate. On the other hand, the methods we are using are applicable also to non-supersymmetric theories with several flavors of adjoint fermions. In addition to that, we can use these methods to study non-constant correlation functions that determine the spectrum of the theory.

The paper is organized as follows. In Sec. II we discuss some general aspects of chiral symmetry breaking in theories with fermions in the adjoint representation. In Sec. III we describe the structure of the instanton zero mode wave functions and in IV we calculate matrix elements of the Dirac operator between zero mode states. These results are used in order to determine the fermion determinant in the field of an instanton–anti-instanton pair (Sec. V) and to calculate the gluino condensate in a random instanton ensemble (Sec. VI). In Sec. VII we describe simulations of an interacting instanton ensemble with different numbers of fermions in the fundamental and adjoint representation.

## II. QCD WITH ADJOINT FERMIONS

QCD with adjoint fermions is defined by the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N_{ad}} \frac{1}{2} \bar{\lambda}_M^{(i)a} (i \not{D})^{ab} \lambda_M^{(i)b} - \frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a, \quad (3)$$

where  $\lambda_M^a$  is a Majorana fermion in the adjoint representation of the gauge group and  $G_{\mu\nu}^a$  is the usual field strength tensor. The covariant derivative in the adjoint representation is given by

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + f^{abc} A_\mu^c. \quad (4)$$

For several Majorana flavors the theory (3) possesses a  $SU(N_{ad})$  chiral symmetry. A non-zero gluino condensate

$$\langle \bar{\lambda}_M^{(i)} \lambda_M^{(j)} \rangle = -\delta^{ij} \sigma \quad (5)$$

breaks this symmetry to  $SO(N_{ad})$  [24]. This fact can be seen most easily by considering the conserved vector and axial-vector currents [25]. There are  $\frac{1}{2}N_{ad}(N_{ad}-1)$  conserved vector currents  $V_\mu^{ij} = \bar{\lambda}_M^{(i)} \gamma_\mu \lambda_M^{(j)}$  and  $\frac{1}{2}N_{ad}(N_{ad}+1)$  classically conserved axial-vector currents  $A_\mu^{ij} = \bar{\lambda}_M^{(i)} \gamma_\mu \gamma_5 \lambda_M^{(j)}$ . The singlet axial current  $A_\mu^{ii}$  is anomalous. At the quantum level this leaves  $N_{ad}^2 - 1$  conserved charges that generate the  $SU(N_{ad})$  chiral symmetry. Gluino condensation breaks the axial symmetries and leads to the appearance of  $\frac{1}{2}(N_{ad}^2 + N_{ad} - 2)$  Goldstone bosons. The unbroken  $\frac{1}{2}N_{ad}(N_{ad}-1)$  vector charges generate the residual  $O(N_{ad})$  symmetry.

In the case of supersymmetric gluodynamics,  $N_{ad}=1$ , there is no continuous symmetry. Instantons break the axial  $U(1)_A$  symmetry but leave a discrete  $Z_{N_c}$  symmetry intact.

This discrete symmetry is spontaneously broken by gluino condensation. As discussed above, the value of the gluino condensate is known from a weak coupling instanton calculation. There are no predictions for the spectrum of the theory, but we expect the lowest states to fill out a chiral supermultiplet containing a scalar and a pseudoscalar meson as well as a Majorana fermion. These results can be summarized in terms of an effective Lagrangian [26]. This is not an effective Lagrangian in the Wilsonian sense. The effective action does not generate the low momentum scattering amplitudes of the theory. Instead, it mainly serves as a generating functional for the anomalous Ward identities of the theory.

## III. INSTANTON GAUGE POTENTIAL AND FERMIONIC ZERO MODES

In theories with adjoint fermions it is convenient to employ a spinor notation for spin, vector, and color indices [8]. We can convert vectors to spinors using

$$V_{\alpha\dot{\alpha}} = V_\mu (\sigma_\mu)_{\alpha\dot{\alpha}}. \quad (6)$$

The Euclidean spinor conventions used in this paper are summarized in Appendix A. The instanton gauge potential couples spin to color degrees of freedom. A field  $A^a$  in the adjoint representation of  $SU(2)$  can be represented by a symmetric tensor  $A^{\alpha\beta}$

$$A^a = A^{\alpha\beta} \epsilon_{\alpha\gamma} (\tau^a)_{\beta}^{\gamma}. \quad (7)$$

In spinor notation, the instanton gauge potential in regular gauge is given by

$$A_{\gamma\delta}^{\alpha\beta} = -2i (\delta_\gamma^\alpha x_\delta^\beta + \delta_\gamma^\beta x_\delta^\alpha) \frac{1}{x^2 + \rho^2}. \quad (8)$$

We can transform the gauge potential to singular gauge using the gauge transformation

$$U^{\dot{\alpha}\alpha} = \hat{x}_\mu (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}. \quad (9)$$

Note that this matrix transforms an undotted color index into a dotted one. We can perform a “fake” conversion of the dotted spinor back to an undotted one using the fact that  $(\sigma_0)^{\alpha\dot{\alpha}}$  is just the unit matrix.

In the case of  $SU(2)$ , the Dirac operator in the background field of an instanton has four zero modes. The first two are conventionally called the supersymmetric (ss) zero modes [13]

$$\lambda_{\alpha(\beta)}^{\gamma\delta} = (\delta_\alpha^\gamma x_\beta^\delta + \delta_\beta^\gamma x_\alpha^\delta) \frac{\rho^2}{\pi (x^2 + \rho^2)^2}, \quad (10)$$

where  $\beta=1,2$  enumerates the zero modes. The other two are referred to as the superconformal (sc) zero modes

$$\lambda_{\alpha(\beta)}^{\gamma\delta} = (\delta_\alpha^\gamma x_\beta^\delta + \delta_\alpha^\delta x_\beta^\gamma) \frac{\rho}{\sqrt{2}\pi (x^2 + \rho^2)^2}. \quad (11)$$

In singular gauge, the zero modes are given by

$$\lambda_{\alpha(\beta)}^{\dot{\gamma}\delta} = (x_{\alpha}^{\dot{\gamma}} x_{\beta}^{\delta} + x_{\beta}^{\dot{\gamma}} x_{\alpha}^{\delta}) \frac{\rho^2}{\pi} \frac{1}{x^2(x^2 + \rho^2)^2} (ss), \quad (12)$$

$$\lambda_{\alpha(\beta)}^{\dot{\gamma}\delta} = (x_{\alpha}^{\dot{\gamma}} \delta_{\beta}^{\delta} + x_{\alpha}^{\delta} \delta_{\beta}^{\dot{\gamma}}) \frac{\rho}{\sqrt{2}\pi} \frac{1}{(x^2 + \rho^2)^2} (sc). \quad (13)$$

Analogously, we can construct the zero modes of the Dirac operator in the background field of an anti-instanton. The regular gauge supersymmetric zero mode has the structure  $\lambda_{\dot{\gamma}\delta}^{\alpha(\beta)} \sim (\delta_{\dot{\gamma}}^{\alpha} \delta_{\delta}^{\beta} + \delta_{\delta}^{\alpha} \delta_{\dot{\gamma}}^{\beta})$ , etc.

The effect of the zero modes on the propagation of fermions can be summarized in terms of an effective Lagrangian [27]. The 't Hooft effective interaction in the case of one Majorana fermion in the adjoint representation of  $SU(2)$  was determined in [28,29]. The result is

$$\begin{aligned} \mathcal{L} = & \frac{4\pi^4}{3} \left( \frac{2\pi}{\alpha_s} \right)^4 \exp\left( -\frac{2\pi}{\alpha_s} \right) \rho^3 d\rho \\ & \times \left\{ \bar{\lambda}_M^a \lambda_M^a \partial_{\mu} \bar{\lambda}_M^b \partial^{\mu} \lambda_M^b + \bar{\lambda}_M^a \gamma_5 \lambda_M^a \partial_{\mu} \bar{\lambda}_M^b \gamma_5 \partial^{\mu} \lambda_M^b \right. \\ & \left. - \frac{1}{2} \bar{\lambda}_M^a \sigma_{\alpha\beta} \lambda_M^b \partial_{\mu} \bar{\lambda}_M^{\alpha\beta} \sigma^{\alpha\beta} \partial^{\mu} \lambda_M^a \right\}. \end{aligned} \quad (14)$$

This result has to be interpreted with some care. The notion of an effective interaction induced by instantons of some fixed size is incompatible with supersymmetry. In order to derive manifestly supersymmetric results we always have to integrate over the collective coordinates of the instanton. Nevertheless, it is instructive to compare the result (14) with the effective interaction in the case of  $N_f=2$  Dirac fermions in the fundamental representation. The structure of the two interactions is quite similar, which suggests that instantons may lead to similar physical effects. The most important difference between the two effective Lagrangians is the presence of derivatives acting on two of the four Majorana spinors in (14). This difference is connected with the asymptotic behavior of the supersymmetric zero modes, which is not  $\sim 1/z^3$ , but  $1/z^4$ .

#### IV. MATRIX ELEMENTS OF THE DIRAC OPERATOR

In the following, we wish to study the spectrum of the Dirac operator in an instanton ensemble. For this purpose, we have to calculate matrix elements of the Dirac operator between the zero modes of individual instantons and anti-instantons

$$T_{IA} = \int d^4x \bar{\lambda}_I^a (i\mathcal{D})^{ab} \lambda_A^b. \quad (15)$$

An ensemble of instantons and anti-instantons is only an approximate saddle point of the action. If the system is sufficiently dilute then the instantons and anti-instantons are well separated and the approximate saddle point solution for the gauge potential is given by a simple sum of the gauge

potentials of the individual instantons. For this purpose, the gauge potential of the individual instantons has to be put in singular gauge. In the sum ansatz, we can use the equations of motion of the fermion fields in order to replace the covariant derivative in Eq. (15) by an ordinary derivative

$$T_{IA} = - \int d^4x \bar{\lambda}_I^a (i\partial) \lambda_A^a. \quad (16)$$

The structure of the Dirac operator is dictated by the form of the zero modes. In the background field of an instanton–anti-instanton pair we have

$$T_{IA} = \begin{pmatrix} 0 & T_{IA}^{ss-ss} & T_{IA}^{ss-sc} \\ T_{IA}^{ss-ss} & T_{IA}^{ss-sc} & 0 \\ T_{IA}^{sc-ss} & T_{IA}^{sc-sc} & 0 \end{pmatrix}, \quad (17)$$

where the matrix elements  $T_{AI}^{ss}, \dots$  are real quaternions. These quaternions can be decomposed as

$$(T_{AI}^{ss-ss})_{\dot{\beta}\beta'} = T_{\mu}^{ss} (\sigma_{\mu})_{\dot{\beta}\beta'} \quad (18)$$

$$(T_{AI}^{sc-ss})_{\beta\dot{\beta}'} = T_{\mu}^{sc} (\sigma_{\mu})_{\beta\dot{\beta}'} \quad (19)$$

$$(T_{AI}^{ss-sc})_{\dot{\beta}\beta'} = T_{\mu}^{ss-sc} \epsilon_{\dot{\beta}\beta'} + T_{\mu\nu}^{ss-sc} \epsilon_{\dot{\beta}\gamma} (\bar{\sigma}_{\mu\nu})_{\dot{\gamma}\beta'}, \quad (20)$$

$$(T_{AI}^{sc-sc})_{\beta\beta'} = T_{\mu}^{sc-sc} \epsilon_{\beta\beta'} + T_{\mu\nu}^{sc-sc} (\sigma_{\mu\nu})_{\beta\gamma} \epsilon_{\gamma\beta'}. \quad (21)$$

Here,  $T_{\mu}^{ss}$  and  $T_{\mu}^{sc}$  are real vectors,  $T_{\mu}^{ss-sc}$  and  $T_{\mu}^{sc-ss}$  are real scalars, and  $T_{\mu\nu}^{ss-sc}$  and  $T_{\mu\nu}^{sc-ss}$  are self-dual and anti-self-dual tensors, respectively. Chiral symmetry implies that the diagonal blocks of  $T_{IA}$  are zero. The upper right and lower left blocks are related by Hermitian conjugation. For example, we find that

$$(T_{IA}^{ss-ss})_{\beta\dot{\beta}'} = T_{\mu}^{ss} (\bar{\sigma}^{\mu})^{\dot{\beta}\beta'}. \quad (22)$$

In general, we have  $(T^{\dagger})_{AI} = (T)_{IA}$ . The eigenvalues of (17) come in quartets  $(\xi, \xi, -\xi, -\xi)$ . These results are in agreement with the general arguments presented in [25,30].

The functions  $T_{\mu}^{ss}, \dots$  depend on the collective coordinates of the instanton and anti-instanton. We will characterize the relative color orientation by the four vector  $u_{\mu} = 1/2 \cdot \text{tr}(U_A^{\dagger} U_I \sigma_{\mu})$ . Here,  $U_{I,A}$  are  $SU(N_c)$  matrices that characterize the color orientation of the instanton and anti-instanton. For color  $SU(2)$   $u_{\mu}$  is a real vector with  $u^2 = 1$ . Using rotational symmetry and the fact that  $T_{AI}$  is quadratic in  $u_{\mu}$  we have

$$T_{\mu}^{ss} = \hat{z}_{\mu} T_1^{ss} + u_{\mu} (u \cdot \hat{z}) T_2^{ss} + \hat{z}_{\mu} (u \cdot \hat{z})^2 T_3^{ss}, \quad (23)$$

$$T_{\mu}^{sc} = \hat{z}_{\mu} T_1^{sc} + u_{\mu} (u \cdot \hat{z}) T_2^{sc} + \hat{z}_{\mu} (u \cdot \hat{z})^2 T_3^{sc}, \quad (24)$$

$$T^{ss-sc} = T_1^{ss-sc} + T_2^{ss-sc}(u \cdot \hat{z})^2, \quad (25)$$

$$T_{\mu\nu}^{ss-sc} = (u_\mu \hat{z}_\nu - u_\nu \hat{z}_\mu)(u \cdot \hat{z}) T_3^{ss-sc}, \quad (26)$$

where  $z_\mu = z_\mu^A - z_\mu^I$  and the functions  $T_1^{ss}, \dots$  depend on  $(|z_\mu|, \rho_I, \rho_A)$ . For simplicity, we will assume that the dependence on  $\rho_{I,A}$  only enters through their geometric mean  $\bar{\rho} = \sqrt{\rho_I \rho_A}$ . The fact that this assumption is valid to fairly good accuracy was checked in the case of fundamental fermions. In a more sophisticated treatment of the instanton–anti-instanton gauge configuration the dependence of the overlap matrix element on  $(z, \rho_I, \rho_A)$  is restricted by conformal invariance [31].

In the following we shall outline the calculation of the invariant functions  $T_1^{ss}, \dots$ . We describe the case  $T_1^{ss}$  in some detail but relegate the results for the other functions to appendix B. Using the expression (12) for the wave function of the supersymmetric zero modes in singular gauge we find

$$\begin{aligned} T_\eta^{ss} = & \{ \text{tr}(\bar{\sigma}_\mu \sigma_\rho \bar{\sigma}_\beta \sigma_\alpha) \text{tr}(\bar{\sigma}_\nu \sigma_\sigma \bar{\sigma}_\gamma \sigma_\eta) \\ & + \text{tr}(\bar{\sigma}_\mu \sigma_\rho \bar{\sigma}_\beta \sigma_\eta \bar{\sigma}_\nu \sigma_\sigma \bar{\sigma}_\gamma \sigma_\alpha) \} \cdot u_\rho u_\sigma \\ & \times \int d^4x \phi_{\mu\nu}(x-z) \partial_\alpha \phi_{\beta\gamma}(x), \end{aligned} \quad (27)$$

where  $\phi_{\mu\nu}(x)$  is the profile function of the supersymmetric zero mode

$$\phi_{\mu\nu}(x) = \frac{\rho^2}{\pi} \frac{\hat{x}_\mu \hat{x}_\nu}{(x^2 + \rho^2)^2}. \quad (28)$$

The integral in Eq. (27) is most easily calculated in Fourier space. The Fourier transform of  $\phi_{\mu\nu}$  is given by

$$\phi_{\mu\nu}(k) = \delta_{\mu\nu} \phi_1(k) + \hat{k}_\mu \hat{k}_\nu \phi_2(k) \quad (29)$$

with

$$\phi_1(k) = \frac{2\pi\rho^2}{y} \left\{ \frac{4}{y^3} - \left( \frac{4}{y^2} + 1 \right) K_1(y) - \frac{2}{y} K_0(y) \right\}, \quad (30)$$

$$\phi_2(k) = -2\pi\rho^2 \left\{ \frac{16}{y^4} - \left( \frac{16}{y^3} + \frac{4}{y} \right) K_1(y) - \left( \frac{8}{y^2} + 1 \right) K_0(y) \right\}, \quad (31)$$

and  $y = k\rho$ .  $K_n(y)$  is the modified Bessel function of the first kind and order  $n$ . We can now calculate the overlap integral and perform the traces. In momentum space the result is given by

$$T_\eta^{ss}(k) = (-i) [-2\hat{k}_\eta - 8u_\eta(u \cdot \hat{k}) + 16(u \cdot \hat{k})^2] |\phi_2(k)|^2. \quad (32)$$

Finally, we can determine the functions  $T_{1,2,3}^{ss}$  by performing the inverse Fourier transform. In the  $d^4k$  integral all integrations except for the one over the absolute magnitude of  $k$  can be performed analytically. We find

$$T_1^{ss}(z) = \frac{1}{8\pi^2} \int dk \left\{ 2k^4 j_1(kz) - 16 \frac{k^3}{z} j_2(kz) \right\} |\phi_2(k)|^2, \quad (33)$$

$$T_2^{ss}(z) = \frac{1}{8\pi^2} \int dk \left\{ 8k^4 j_1(kz) - 32 \frac{k^3}{z} j_2(kz) \right\} |\phi_2(k)|^2, \quad (34)$$

$$T_3^{ss}(z) = \frac{1}{8\pi^2} \int dk 16k^4 j_3(kz) |\phi_2(k)|^2, \quad (35)$$

where  $j_n(x)$  is the spherical Bessel function of order  $n$ . The integrals (33) have to be performed numerically. The results are shown in Fig. 1. In the following, we will use a simple parametrization of the numerical results. In the case of the supersymmetric overlaps, we use

$$\bar{\rho} T_1^{ss}(z) = \frac{-1.26\bar{z}}{1.0 + 2.34\bar{z}^2 + 0.35\bar{z}^4 + 0.24\bar{z}^6}, \quad (36)$$

$$\bar{\rho} T_2^{ss}(z) = \frac{1.05\bar{z}}{(1.0 + 0.38\bar{z}^2)^3} + \frac{-6.36\bar{z}^3}{(1.0 + 0.68\bar{z}^2)^4}, \quad (37)$$

$$\bar{\rho} T_3^{ss}(z) = \frac{15.8\bar{z}^3}{(1.0 + 0.84\bar{z}^2)^4}, \quad (38)$$

where  $\bar{z} = z/\bar{\rho}$ . These parametrizations respect the asymptotic behavior of the overlap integrals. In particular, we have  $T^{ss}(z) \sim 1/z^5$ ,  $T^{sc}(z) \sim 1/z^3$  and  $T^{ss-sc}(z) \sim 1/z^4$ .

For completeness, let us compare these results to the corresponding expressions in the case of fundamental fermions. In this case, there is only one zero mode per instanton. The overlap matrix element  $T_{IA}$  is a real number in the case of  $SU(2)$ , and complex for  $SU(N_c > 2)$ .  $T_{IA}$  satisfies the symmetry relation  $T_{IA} = T_{AI}^*$ . As a consequence, the eigenvalues are real and occur in pairs  $(\xi, -\xi)$ . We can extract the dependence of  $T_{IA}$  on the collective coordinates. The result is

$$\begin{aligned} T_{AI}^{fund} &= (u \cdot \hat{z}) T^f(z, \rho_I, \rho_A), \\ T^f(z, \rho_I, \rho_A) &\simeq \frac{1}{\rho_I \rho_A} \frac{4z}{[2.0 + z^2/(\rho_I \rho_A)]^2}. \end{aligned} \quad (39)$$

We note that the fundamental overlap matrix element only depends on one  $SU(2)$  angle  $\cos \theta \equiv (u \cdot \hat{z})$ . From the asymptotic form of the zero mode solution one finds  $T^f(z) \sim 1/z^3$ .

## V. THE FERMION DETERMINANT IN THE FIELD OF AN INSTANTON–ANTI-INSTANTON PAIR

Before we study gluino condensation we would like to make a brief digression and discuss the gluino induced instanton–anti-instanton interaction. This interaction will play an important role in the calculation of the gluino con-



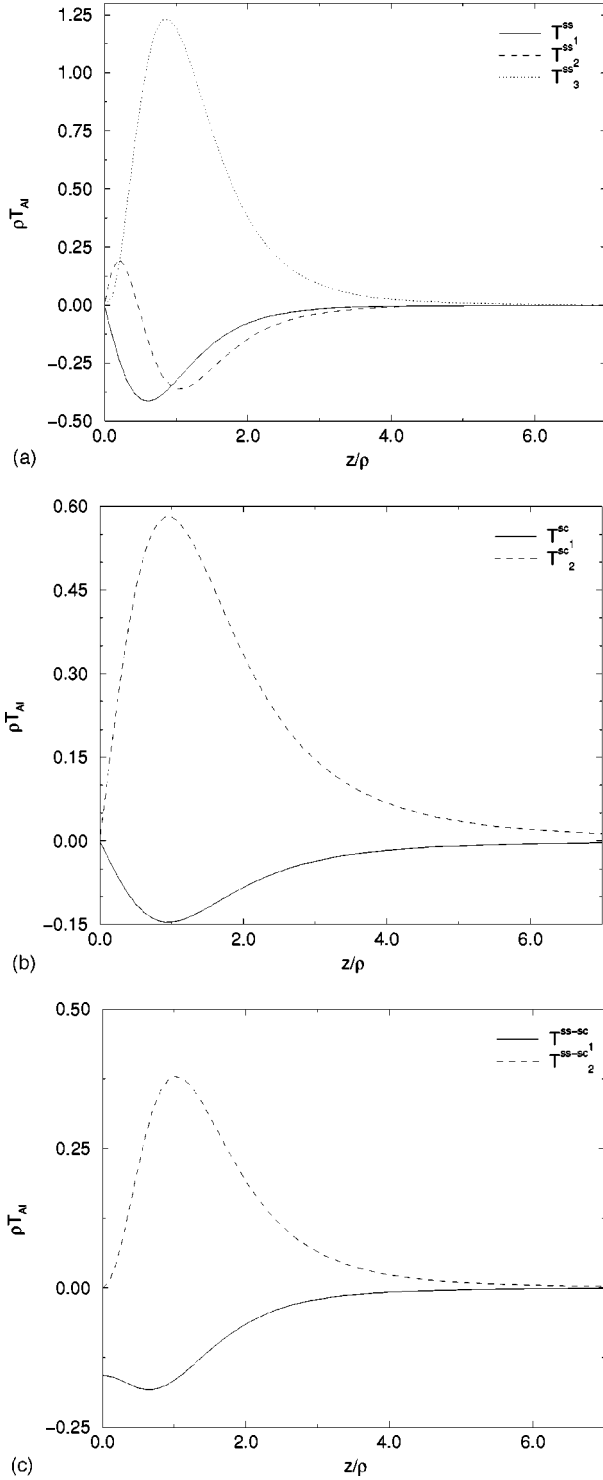


FIG. 1. Invariant functions characterizing the overlap matrix elements of the Dirac operator. (a), (b), and (c) show the diagonal overlap matrix elements between supersymmetric and superconformal zero modes, and the mixed supersymmetric-superconformal matrix elements.

densate in an interacting instanton ensemble.

The probability to find an instanton–anti-instanton pair characterized by the collective coordinates  $(z_{I,A}, \rho_{I,A}, U_{I,A})$  is controlled by the weight factor  $\exp(-S)\det(\mathcal{D}+m)$ . The first factor is the well known gluonic interaction. If the in-

stanton and anti-instanton are well separated it has the dipole form [32]

$$S = 2S_0 - S_0 \frac{4\rho_I^2 \rho_A^2}{z^4} (1 - 4\cos^2 \theta), \quad (40)$$

where  $S_0 = (8\pi^2)/g^2$  is the single instanton action and  $\cos \theta$  is the  $SU(2)$  angle introduced above. We note that the interaction is attractive if the color orientation is aligned with the spatial orientation,  $\cos \theta = \pm 1$ . The second factor is the fermion determinant. In the case of fundamental fermions, it is also well known. We have

$$\det(\mathcal{D}) = \cos^2 \theta \frac{16}{\rho_I^2 \rho_A^2} \frac{z^2}{[2.0 + z^2/(\rho_I \rho_A)]^4}, \quad (41)$$

which is also attractive for  $\cos \theta = \pm 1$ . We also note that the interaction peaks at  $z^2 \approx \rho_I \rho_A$ .

Using the results of the last section we can calculate the fermion induced interaction with fermions in the adjoint representation. In order to calculate the determinant for one Majorana fermion we take the square root of the corresponding expression for a Dirac fermion in the adjoint representation. For simplicity, let us begin with the determinant in the basis of the supersymmetric zero modes only. We find

$$\det(\mathcal{D})^{ss} = |(T_1^{ss})^2 + [(T_2^{ss})^2 + 2T_1^{ss}T_2^{ss} + 2T_1^{ss}T_3^{ss}] \cos^2 \theta + [(T_3^{ss})^2 + 2T_2^{ss}T_3^{ss}] \cos^4 \theta|. \quad (42)$$

The result for the superconformal zero modes is even more simple,

$$\det(\mathcal{D})^{sc} = |(T_1^{sc})^2 + [(T_2^{sc})^2 + 2T_1^{sc}T_2^{sc}] \cos^2 \theta|. \quad (43)$$

This expression is quite similar to the determinant for fundamental fermions. The supersymmetric determinant (42) is somewhat more complicated, but also peaked for  $\cos \theta = \pm 1$ . When the mixing between supersymmetric and superconformal zero modes is included the fermion determinant depends on other  $SU(2)$  angles in addition to  $\cos \theta$ . We show numerical results for  $\log(\det(\mathcal{D}))$  as a function of  $z$ ,  $\cos \theta$ , and  $\cos \phi$  in Fig. 2. Here we have taken  $\hat{z}_\mu = z \delta_{\mu 4}$  and defined  $\cos \theta = u_4$  and  $\sin \theta \cos \phi = u_2$ . We observe that again the determinant peaks for  $z^2 \approx \rho_I \rho_A$ . For large  $z$ , the determinant behaves as  $\sim 1/z^{16}$ . More importantly, we find that the interaction is again most attractive for  $\cos \theta = \pm 1$ . There is some dependence on  $\cos \phi$ , but it is not as pronounced as the dependence on  $\cos \theta$ . This means that the gluino induced interaction for one Majorana fermion is qualitatively similar to the quark induced interaction with an effective number of quark flavors between  $N_f = 2$  (which gives  $\det \sim 1/z^{12}$ ) and  $N_f = 3$  (corresponding to  $\det \sim 1/z^{18}$ ).

## VI. GLUINO CONDENSATION IN A RANDOM INSTANTON ENSEMBLE

In this section we study gluino condensation in a random instanton ensemble. This means that we will assume that the

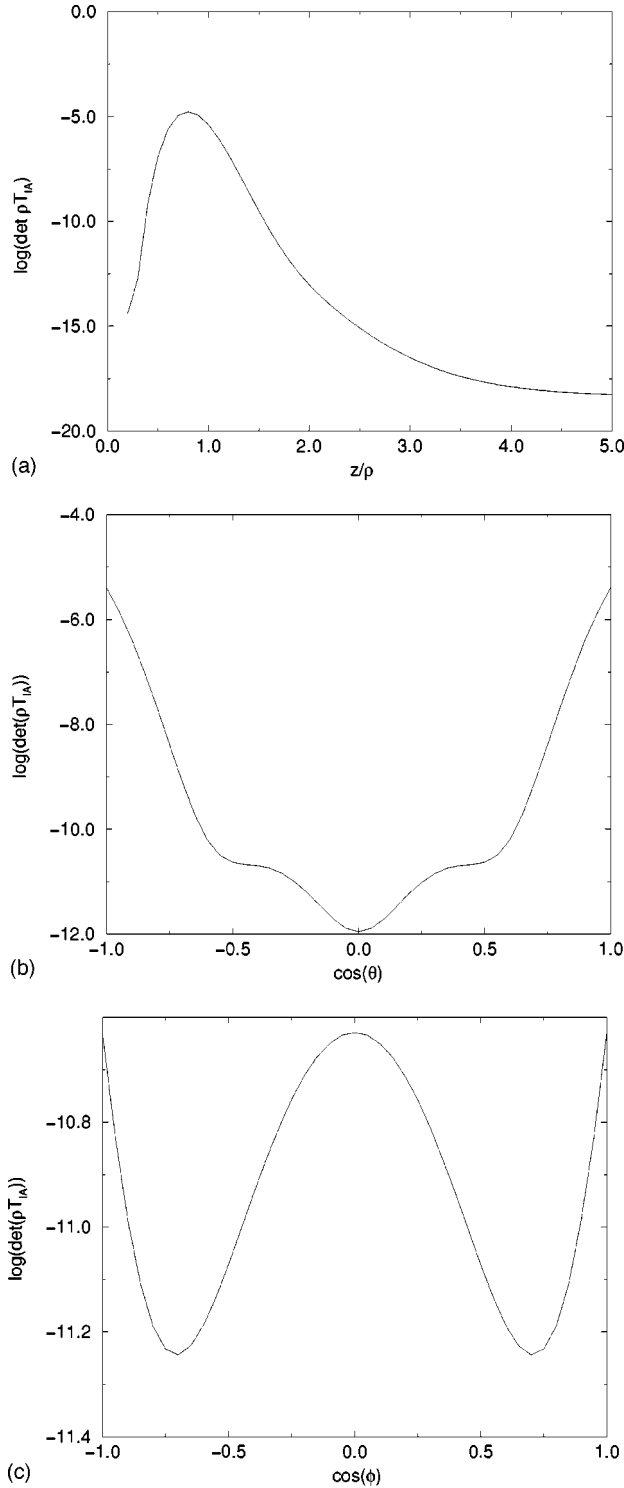


FIG. 2. Logarithm of the adjoint fermion determinant in the field of an instanton–anti-instanton pair. (a) shows  $\log(\det(\mathcal{D}))$  as a function of  $z$  (in units of  $\rho$ ) for  $\cos \theta = 1$ , (b) gives the dependence on  $\cos \theta$  for  $z = 1$ , and (c) the behavior of the determinant as a function of  $\cos \phi$  for  $\cos \theta = 1/2$  and  $z = 1$ .

collective coordinates of the instantons and anti-instanton are distributed randomly. In particular, we shall neglect the effect of the fermion determinant on the distribution of instantons. This is not a good approximation even in ordinary QCD

and it certainly cannot be correct in a supersymmetric theory. Nevertheless, using the approximation of randomness we can get some analytic understanding of the dependence of the gluino condensate on the parameters characterizing the instanton liquid. We can also get an estimate of the relative size of the quark and gluino condensates in theories with both fundamental and adjoint fermions.

The simplest model of the spectrum of the Dirac operator is based on the assumption that the non-zero matrix elements of the Dirac operator are Gaussian random numbers [19]. The distribution is characterized by the first moment

$$\sigma^2 = \left\langle \frac{2}{N} \text{tr}(T^\dagger T) \right\rangle. \quad (44)$$

The eigenvalue distribution for the Gaussian ensemble is given by a semi-circle where the density of eigenvalues at zero virtuality is  $\rho(0) = (N/V)(\pi\sigma)^{-1}$ . Here,  $(N/V)$  is the number of eigenstates per unit volume. The first moment of the overlap matrix can be estimated by averaging  $|T_{AI}|^2$  over the collective coordinates of the instantons. Using (39) we find the first moment of the Dirac operator for fermions in the fundamental representation of  $SU(2)$  [19]

$$\sigma = \left( \frac{1}{3} \frac{N}{V} \right)^{1/2} \bar{\rho} \pi, \quad (45)$$

where  $\bar{\rho}$  is the average size of the instanton. This parameter, just like the density of instantons, cannot be determined in the semi-classical approximation. In the instanton liquid model of the QCD vacuum it is assumed that  $\bar{\rho} = 1/3 \text{ fm}$  and  $(N/V) = 1 \text{ fm}^4$  [18]. Using these values we find

$$\langle \bar{q}q \rangle = -\frac{1}{\pi\bar{\rho}} 3^{1/2} \left( \frac{N}{V} \right)^{1/2} \approx -(230 \text{ MeV})^3, \quad (46)$$

in very good agreement with the phenomenological value [which, of course, applies to color  $SU(3)$ ].

The same arguments can be applied to gluino condensation in a random instanton ensemble. In this case we need to determine the first moment of a quaternionic matrix with the matrix elements determined in Sec. IV. We find

$$\sigma_{ad} = \left( \frac{N}{V} \right)^{1/2} 0.43 \bar{\rho} \pi, \quad (47)$$

which is somewhat smaller than Eq. (45). There are four times as many eigenstates per unit volume but for a Majorana fermion the Banks-Casher relation has an additional factor  $1/2$ ,  $\langle \bar{\lambda}\lambda \rangle = -\pi/2 \cdot \rho(0)$ . We finally get the following estimate of the gluino condensate:

$$\langle \bar{\lambda}\lambda \rangle = -\frac{1}{\pi\bar{\rho}} 4.6 \left( \frac{N}{V} \right)^{1/2}. \quad (48)$$

Here and in what follows we have dropped the subscript  $M$  on the Majorana spinors. This result is a little more than twice as large as the corresponding result for a Dirac fermion

in the fundamental representation. As we saw, this is mainly due to the effective number of zero modes in both cases. We emphasize that the gluino condensate is proportional to the square root of the instanton density, which is also what one would expect if the condensate is extracted from the four-point function using clustering.

We have checked these estimates by performing a numerical calculation of the spectrum of the Dirac operator in a random instanton ensemble. This means that instead of assuming the matrix elements of the Dirac operator to be random we take the collective coordinates of the instantons and anti-instantons to be random. We calculate the spectrum of the Dirac operator and determine the gluino condensate using

$$\langle \bar{\lambda} \lambda \rangle = -\frac{1}{2} \int d\lambda \rho(\lambda) \frac{2m}{\lambda^2 + m^2}. \quad (49)$$

The results are shown in Fig. 3. We observe that the spectrum is not a semi-circle but is peaked towards zero virtuality [33]. This non-analyticity is smoothed out when we calculate the condensate for a non-zero quark or gluino mass. Again using  $(N/V)=1 \text{ fm}^4$  and  $\bar{\rho}=1/3 \text{ fm}$  as well as  $m_q=m_g=20 \text{ MeV}$  we find  $\langle \bar{\psi} \psi \rangle = -(260 \text{ MeV})^3$  and  $\langle \bar{\lambda} \lambda \rangle = -(347 \text{ MeV})^3$ .

## VII. GLUINO CONDENSATION IN AN UNQUENCHED INSTANTON ENSEMBLE

As we stressed in the previous section, the assumption of randomness is not expected to be very useful. The fermion determinant is given by the product of all eigenvalues of the Dirac operator, while the quark or gluino condensate is determined by the density of small eigenvalues. This implies that the determinant tends to suppress fermion condensates. In particular, we expect that the strength of chiral symmetry breaking is reduced as the number of fermion flavors is increased.

In this section we shall study this problem using simulations of the instanton ensemble in QCD with fundamental and adjoint fermions. We consider the partition function

$$Z = \int \left( \prod_i^N d\Omega_i d(\rho_i) \right) \det(\mathcal{D}_f + m_q)^{N_f} \times \det(\mathcal{D}_a + m_g)^{N_{ad}/2} \exp(-S). \quad (50)$$

Here,  $\Omega$  denotes the collective coordinates of the instanton,  $d(\rho)$  is the single instanton distribution [27,29,34],  $\mathcal{D}_{f,a}$  are the Dirac operators in the fundamental and adjoint representation, and  $\exp(-S)$  is the gluonic interaction between instantons. In order to study spontaneous symmetry breaking in a finite volume we introduce non-zero quark and gluino masses  $m_{q,g}$ . We will study the limit  $m_{q,g} \rightarrow 0$  in some detail.

The partition function (50) suffers from the usual IR problem connected with large instantons for which the semi-classical approximation does not apply. In practice, we deal with this problem by introducing a short range repulsive core

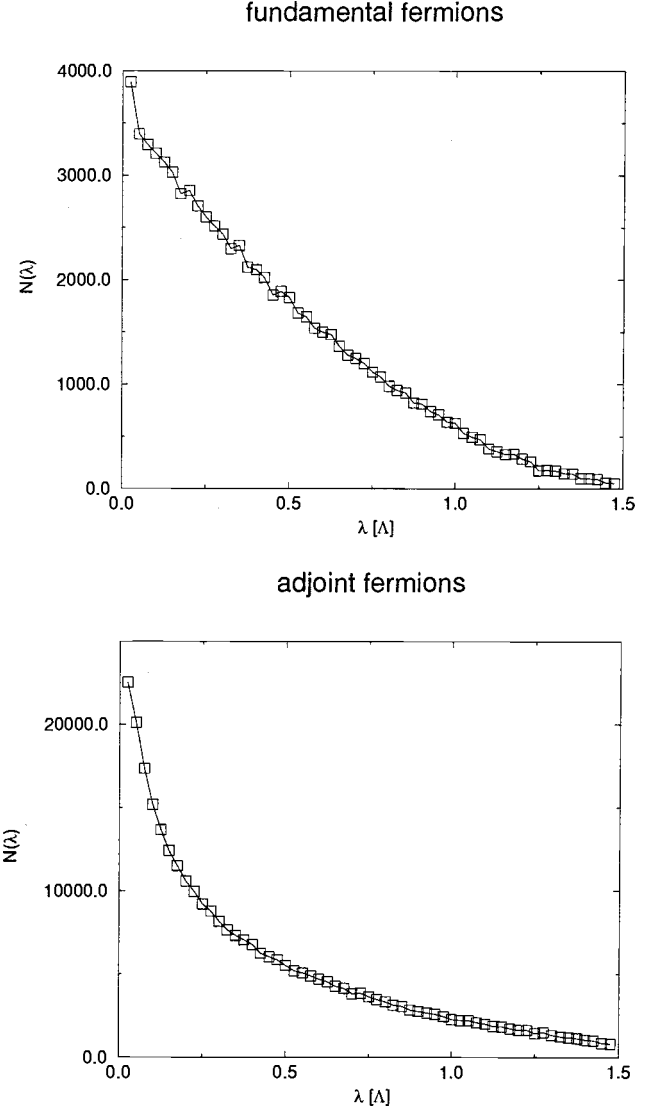


FIG. 3. Spectrum of the fundamental and adjoint Dirac operator in a random instanton ensemble. The spectral density is given in arbitrary units.

in the gluonic instanton interaction; see Sec. V C in [22] for a more detailed discussion. The repulsive core eliminates the contributions of large instantons and very close pairs. This particular method for suppressing objects that are not semi-classical has the virtue that it respects the classical scale invariance of Yang-Mills theory.

The instanton ensemble is characterized by two numbers, the scale parameter  $\Lambda$  that enters into the instanton weight  $d(\rho)$  and a dimensionless parameter  $A$  which determines the size of the core. Lacking a better theory of topological fluctuations beyond the semi-classical domain we have to fix  $A$  phenomenologically. This could be done, for example, as soon as lattice information on the spectrum and other properties of theories with adjoint fermions becomes available [35,36]. In this work we will use the same value that was employed in studies of QCD with fundamental fermions. It leads to a dilute instanton ensemble characterized by the dimensionless parameter  $\bar{\rho}^4(N/V) \approx 0.12$ . For simplicity we

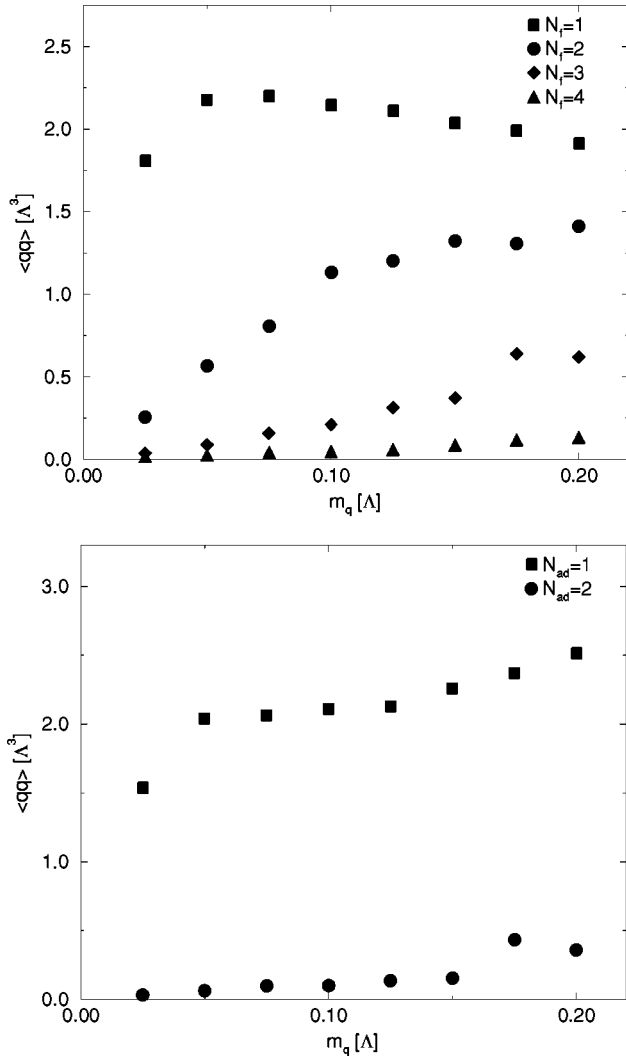


FIG. 4. Quark condensate in an interacting instanton ensemble as a function of quark or gluino mass. (a) shows the quark condensate for  $N_f = 1, \dots, 4$  Dirac fermions in the fundamental representations  $SU(2)$ . (b) shows the gluino condensate for  $N_{ad} = 1, 2$  Majorana fermions in the adjoint representation.

will concentrate on simulations at a fixed instanton density  $(N/V) = 1.0\Lambda^4$ .

To set the stage, we show results for  $N_f = 1, \dots, 4$  flavors of fundamental fermions. Figure 4 shows the quark condensate as a function of the quark mass from simulations in a Euclidean box of size  $V = 2.0^4\Lambda^4$ . The case of only one flavor is special. The chiral condensate persists even if the limit  $m_q \rightarrow 0$  is taken in a finite volume. This is due to the fact that for  $N_f = 1$  the quark condensate is dominated by direct instanton contributions. The result for  $N_f = 2$  is characteristic of spontaneous symmetry breaking. The quark condensate vanishes as the quark mass goes to zero but shows a clear plateau for larger quark masses. One can verify that the onset of chiral symmetry breaking moves towards smaller masses as the volume is increased. For more than two flavors the chiral condensate is significantly reduced. In the case of three flavors the signal is already quite weak. Using simulations in bigger volumes one can verify that chiral symmetry is indeed

broken. There is no clear evidence for chiral symmetry breaking in simulations with four or more flavors.

Figure 4(b) shows the gluino condensate measured in simulations with one or two flavors of Majorana fermions in the adjoint representation. For  $N_{ad} = 1$  there is clear evidence for spontaneous symmetry breaking. Indeed, the behavior is more reminiscent of the case  $N_f = 1$ , where  $\langle \bar{q}q \rangle$  receives direct instanton contributions, than the case  $N_f = 2$ , in which chiral symmetry breaking is a collective effect.

These observations can be understood in more physical terms. Supersymmetric gluodynamics has no Goldstone bosons, so finite volume effects are much weaker than in  $N_f = 2$  non-supersymmetric QCD. This means that in a fixed volume, gluino condensation can be observed for gluino masses that are significantly smaller than the quark masses required to produce quark condensation. In the standard picture, there is a discrete chiral symmetry which is broken by gluino condensation. This means that if the gluino mass is too small then chiral symmetry will be restored because of tunneling between the  $Z_2$  vacua. This is different from  $N_f = 1$  non-supersymmetric QCD where instantons leave no unbroken discrete symmetries.

The value of the gluino condensate is  $\langle \bar{\lambda}\lambda \rangle \approx 2\Lambda^3$ . This result has the correct order of magnitude but it cannot yet be compared directly to the prediction (1). First of all, we use a different definition of the scale parameter. In order to make a connection with our work on QCD we use a Pauli-Villars scale parameter. Second, we have an additional parameter  $A$  which controls the boundary of the semi-classical regime. Finally, we have performed the simulations at a fixed density of instantons  $(N/V) = 1.0\Lambda^4$ . It is this choice which effectively sets the scale in our calculation.

In Fig. 4(b) we also show the gluino condensate measured in simulations with  $N_{ad} = 2$  Majorana flavors. The condensate is very small and there is no clear evidence for spontaneous chiral symmetry breaking.

The spectrum of the Dirac operator for  $N_f = 2$  quark flavors and  $N_{ad} = 1$  Majorana flavor is shown in Figs. 5(a) and 5(b). The spectra were determined in simulations with  $m_{q,g} = 0.1\Lambda^{-1}$ . Again, we observe that in both cases there is a finite density of eigenvalues as  $\lambda \rightarrow 0$ . For  $N_f = 2$  the spectral density near  $\lambda = 0$  is flat,<sup>1</sup> whereas in the case  $N_{ad} = 1$  it is growing towards small  $\lambda$ . Again, this is similar to the case of only one fundamental fermion. The results are consistent with the effective field theory prediction [37,38]

$$\rho'(\lambda=0) = \frac{\Sigma_0^2}{16\pi^2 f_\pi^4} \frac{(N_f - 2)(N_f + \beta)}{\beta N_f}. \quad (51)$$

Here,  $\beta$  is the Dyson index of the random matrix ensemble with the appropriate symmetry. We have  $\beta = 1$  for funda-

<sup>1</sup>Figure 5 shows that the spectral density is flat for intermediate values of  $\lambda$ . There is a finite volume suppression of the spectral density for small  $\lambda$  and a  $O(m^2)$  peak at  $\lambda = 0$ . To show that the spectral density is flat at  $\lambda = 0$  in the limit  $V \rightarrow \infty, m \rightarrow 0$  requires more numerical work.



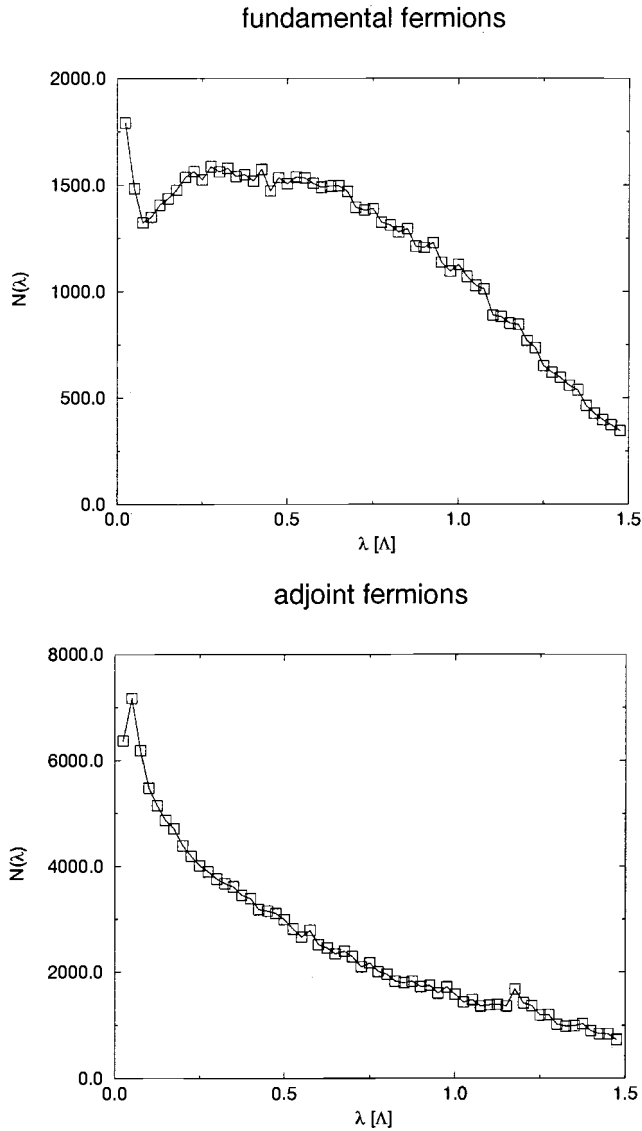


FIG. 5. Spectrum of the fundamental and adjoint Dirac operator in an unquenched instanton ensemble. In the case of the fundamental spectrum the ensemble was created with  $N_f=2$  fundamental Dirac fermions, while in the case of the adjoint spectrum the ensemble corresponds to  $N_{ad}=1$  adjoint Majorana fermions.

mental fermions in  $SU(2)$ ,  $\beta=2$  for fundamental fermions in  $SU(N>2)$ , and  $\beta=4$  in the case of fermions in the adjoint representation.  $N_f$  denotes the number of Dirac or Majorana flavors in the cases  $\beta=1,2$  and  $\beta=4$ , respectively.  $\Sigma_0$  is the magnitude of the quark condensate and  $f_\pi$  the pion decay constant. The expression (51) summarizes the fact that the spectrum is peaked towards small virtuality for both  $N_f=1$  and  $N_{ad}=1$  while it is flat for  $N_f=2$ . Effective field theory predicts the slope of the Dirac spectrum under the assumption that chiral symmetry is broken. The theory cannot predict whether chiral symmetry breaking takes place for some given  $N_f$  or  $N_{ad}$ .

### VIII. CONCLUSIONS

In summary we have studied gluino condensation and the spectrum of the Dirac operator in an instanton ensemble. We

employ the semi-classical approximation and focus on the Dirac operator in the subspace spanned by the zero modes of the individual instantons and anti-instantons. We have shown how the quaternionic structure of the Dirac operator in theories with adjoint fermions emerges naturally from the spin and color structure of the zero modes. The dependence of the matrix elements on the collective coordinates of the instantons is quite complicated but qualitatively similar to the simpler case of fundamental fermions.

We have provided evidence that gluino condensation does take place in an ensemble of instanton and anti-instantons. In a random ensemble, the gluino condensate is proportional to the square root of the instanton density. In supersymmetric gluodynamics we find that gluino condensation persists even if interactions between the instantons are taken into account. We observed that finite volume effects are much weaker than in QCD with two flavors of fundamental fermions. This is consistent with the fact that supersymmetric gluodynamics has a large mass gap. In QCD with more than one adjoint flavor we found no compelling evidence for gluino condensation.

There are many problems that remain to be studied. In particular, it would be interesting to make a systematic study of gluino and gluino-gluon correlation functions. There are two types of correlation functions: constant correlators that provide information on condensates, and  $x$ -dependent correlators related to the spectrum. These correlation functions will also show to what extent supersymmetry is realized in the limit  $m_g \rightarrow 0$ . In addition to that, it would be interesting to search for evidence of  $Z_2$  domains and to investigate the dependence of the results on the topological sector of the theory. In this work we have used the zero mode wave functions that correspond to trivial holonomy and anti-periodic boundary conditions on the fermions. This suggests the question of how the results are changed if the boundary conditions are modified. In this case, the zero modes discussed in [39] will come into play. Finally, it is important to study the role of very large instantons that were excluded in the present study.

There have been suggestions that objects with fractional topological charge may play a role in theories with adjoint fermions [40–42,15]. These objects can give a direct contribution to the gluino condensate. Because of tunneling between the different  $Z_N$  phases the presence of such objects cannot be inferred from the behavior of the gluino condensate as a function of the quark mass in a finite volume. One should be able, however, to detect the presence of fractionally charged objects in lattice simulations by looking for zero modes of the Dirac operator that do not appear in multiples of  $2N_c$  [43]. In this context it would also be interesting to study gluino condensation for  $N_c > 2$ . For adjoint fermions the number of zero modes per topological charge increases with  $N_c$ . One might therefore doubt that instantons alone are sufficient to trigger gluino condensation in large  $N_c$  SUSY gluodynamics. It has also been suggested that fractionally charged objects can be thought of as instanton constituents [44,41,45]. One might then envision a situation where if  $N_c$  is small, or instantons are small, fractionally charged objects are bound into instantons while for large  $N_c$ , or for large

instantons, topological objects dissociate and the instanton liquid should be replaced by liquid of fractional charges.

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### APPENDIX A: EUCLIDEAN SPINOR CONVENTIONS

We use the following Euclidean spinor conventions

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} = \gamma_\mu^\dagger, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A1})$$

$$\sigma_\mu = (i\vec{\sigma}, 1), \quad \bar{\sigma}_\mu = (-i\vec{\sigma}, 1), \quad (\text{A2})$$

$$(\sigma_{\mu\nu})_\alpha^\beta = \frac{1}{4} [(\sigma_\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\nu)^{\dot{\alpha}\beta} - (\sigma_\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}], \quad (\text{A3})$$

$$(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{4} [(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}(\sigma_\nu)_{\alpha\beta} - (\bar{\sigma}_\nu)^{\dot{\alpha}\alpha}(\sigma_\mu)_{\alpha\beta}]. \quad (\text{A4})$$

Indices are raised and lowered with  $\epsilon^{\alpha\beta}$  and  $\epsilon^{\dot{\alpha}\dot{\beta}}$  where  $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta_\gamma^\alpha$  and  $\epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon^{\alpha\beta}$ . The Euclidean sigma matrices have the following properties:

$$(\sigma_\mu \bar{\sigma}_\nu)^\beta_\alpha = \delta_{\mu\nu} \delta_\alpha^\beta + 2(\sigma_{\mu\nu})_\alpha^\beta, \quad (\text{A5})$$

$$(\bar{\sigma}_\mu \sigma_\nu)_{\dot{\beta}}^{\dot{\alpha}} = \delta_{\mu\nu} \delta_{\dot{\alpha}}^{\dot{\beta}} + 2(\bar{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}}, \quad (\text{A6})$$

$$(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma_\mu)_{\beta\dot{\beta}} \quad (\text{A7})$$

$$\sigma_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma}, \quad (\text{A8})$$

$$\bar{\sigma}_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \bar{\sigma}_{\rho\sigma}.$$

### APPENDIX B: MATRIX ELEMENTS

In this appendix we collect the remaining matrix elements of the Dirac operator. We define the profile function of the superconformal zero mode

$$\phi_\mu = \frac{\rho}{\sqrt{2}\pi} \frac{x_\mu}{(x^2 + \rho^2)^2} \quad (\text{B1})$$

and its Fourier transform  $\phi_\mu(k) = -i\hat{k}_\mu \phi_3(k)$  with

$$\phi_3(k) = -\sqrt{2}\pi\rho^2 K_1(k\rho). \quad (\text{B2})$$

The matrix elements of the Dirac operator between superconformal zero modes are determined by

$$T_\eta^{sc}(k) = (+i)[-2\hat{k}_\eta + 8u_\eta(u \cdot \hat{k})^2]k|\phi_3(k)|^2, \quad (\text{B3})$$

and the matrix elements between supersymmetric and superconformal zero modes lead to

$$T^{ss-sc}(k) = [2 - 8(u \cdot \hat{k})^2]k\phi_2(k)\phi_3(k), \quad (\text{B4})$$

$$T_{\mu\nu}^{ss-sc}(k) = 8(u_\mu \hat{k}_\nu - u_\nu \hat{k}_\mu)(u \cdot \hat{k})k\phi_2(k)\phi_3(k). \quad (\text{B5})$$

From these results we can extract the invariant functions

$$T_1^{sc}(z) = -\frac{1}{8\pi^2} \int dk 2k^4 j_1(kz) |\phi_3(k)|^2 \quad (\text{B6})$$

and  $T_2^{sc}(z) = -4T_1^{sc}(z)$  as well as  $T_3^{sc}(z) = 0$ . Also

$$T_1^{ss-sc}(z) = \frac{1}{8\pi^2} \int dk \left[ 2k^4 j_0(kz) - 8\frac{k^3}{z} j_1(kz) \right] \phi_2(k)\phi_3(k), \quad (\text{B7})$$

$$T_2^{ss-sc}(z) = \frac{1}{8\pi^2} \int dk 8k^4 j_2(kz) \phi_2(k)\phi_3(k), \quad (\text{B8})$$

and  $T_3^{ss-sc}(z) = -T_2^{ss-sc}(z)$ . Numerical results for these functions are shown in Fig. 1. The results can be parametrized as

$$\bar{\rho} T_1^{sc}(z) = \frac{-0.25\bar{z}}{1.0 + 0.42\bar{z}^2 + 0.21\bar{z}^4} \quad (\text{B9})$$

as well as

$$\bar{\rho} T_1^{ss-sc}(z) = \frac{-0.17}{1.0 + 0.05\bar{z}^2 + 0.08\bar{z}^4}, \quad (\text{B10})$$

$$\bar{\rho} T_2^{ss-sc}(z) = \frac{1.2\bar{z}^2}{(1.0 + 0.45\bar{z}^2)^3} + \frac{0.014\bar{z}^2}{(1.0 + 0.21\bar{z}^2)^3}, \quad (\text{B11})$$

where  $\bar{z} = z/\bar{\rho}$ . The overlap matrix elements  $T^{ss-sc}$  are related to the corresponding functions with the supersymmetric and superconformal zero modes interchanged. We find  $T_{1,2}^{sc-ss} = -T_{1,2}^{ss-sc}$  and  $T_3^{sc-ss} = T_3^{ss-sc}$ .

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